The Logistic Map: Journey into Chaos

Donna Janzou

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The logistic map is widely used as an introduction to deterministic chaos as it is simple, one-dimensional, discrete equation that produces chaos at certain growth rates. The logistic map iterates, by generation, the change in a population. It is the quintessential example of Robert May’s “Simple Mathematical Models With Very Complicated Dynamics” (May, 1976). In this paper we will use analytical techniques and Pynamical software to study the onset of chaos in the logistic mapping.

The logistic mapping is the iterative version of the differential logistic equation originally used by Pierre-Francois Vernhulst in 1838; Vernhulst adapted Thomas Malithus’ growth model which showed unconstrained exponential growth of a population into a more reality- based model which constrains unlimited growth of a population due to the scarcity of resources. The logistic differential equation is defined as

(1)

Where r represents the growth rate and K represents the carrying capacity of the environment (Wolfram MathWorld, n.d.).

This differential equation (1) is rewritten as a difference equation resulting in a discrete time model for demographics, the logistic mapping. This change is necessary as the logistic mapping shows chaotic behavior in one dimension while the analogous differential equation does not until you consider a two-dimensional map. To keep our example as simple as possible, consider

(2)

Equation (2) is an example of an iterative equation. Given an initial value you can find ) easily by substitution. In this way the logistic mapping represents a completely deterministic formula. One might expect such a simple deterministic system as the logistic mapping would produce predictable behavior.

However, over time, this system can produce wildly unpredictable,

divergent and fractal, infinitely detailed and self-similar without ever actually repeating; this behavior is

due to sensitivity to slight changes in parameter, and initial conditions (Boeing, 2016). In this paper we will be look at the chaos brought about by small changes in the value of

In equation (2), is the growth rate of the population; it is a constant determined empirically for a population; the parameter changes for different populations but the variables and the equation itself do not. Notice that the new equation does not have a K value. The value is now adjusted to include both the reproduction and starvation factors relevant to a population.

Also, instead of P representing the population, represents the present population as a ratio between 0 and 1. For example, this convention, would represent a present population of 600 elk on an area of land known to be able to “carry” 1200 elk as and would be the population of the next iteration (generation) of elk. Restricting to the interval [0,1] is necessary as the logistic mapping would take values of less than 0 or greater than 1 quickly toward .

We will restrict the value of to [0,4]. A graph of the logistic map is shown below; an r value greater than 4 would run to - as shown by the second cobweb plot below; .

Whereas the first cobweb plot for is interesting.

Chart

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To understand some of the basics of chaos theory inherent in the logistic mapping, it is necessary to be aware of some definitions.

In an iterative mapping, a fixed point is a point which does not change with iteration.

A point, x, is fixed if

A point that is fixed is fixed forever which means that applying to the fixed point any number of times (iterating any number of times) will always return the same value.

For our logistic mapping, the fixed points can be found by finding the values for which iteration does not change the value, that is, where

These can easily be found by solving the system of equations:

, using the quadratic equation, we get

The fixed points can be seen graphically as the points where the lines intersect. Notice that, if the parabola does not “reach” the line so the only point of intersection is the fixed point The fixed point is born when the value of the parameter .

Chart, line chart

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When is a non- trivial fixed point. is a trivial fixed point as it represents a population of 0 on which no growth rate can have any effect; the population remains 0.

Fixed points are attracting, repelling or indifferent. An attracting fixed point is stable while a repelling fixed point is unstable. To test the stability of a fixed point, you look at whether points in the vicinity of the fixed point are attracted to the fixed point; we are essentially looking at whether iteration brings a nearby point closer to the fixed point (attracted) or further away (repelled).

look at the derivative of the function at the fixed point f , then the fixed point is stable and attracting. If the fixed point is indifferent (neither attracting nor repelling) and if the fixed point is unstable and repelling.

The motivation for this definition follows:

Consider a point very close to the fixed point , say . If is very small then we can say that

Then . If is closer to the fixed point after the iteration meaning that nearby points are attracted to the fixed point; the fixed point is attracting (Brockman, 2018). An attractor is the value or set of values that a system settles on over time.

We can Use the derivative test at the fixed points we found to determine where the fixed points

and are attracting on the logistic mapping, For ease of notation and computation, will be written .

. Find the values of for which the fixed point is attracting.

, so when

As is restricted to the interval [0,4], we have that 0 is an attracting fixed point whenis on the interval [0,1).

Find the values of for which the fixed point is attracting.

, therefore, is an attracting fixed point when

Notice that is indifferent.

In particular, the behavior of the system is related to the derivative of at the non-trivial fixed point . When , )| > 1 the fixed point becomes unstable and a bifurcation occurs. A bifurcation happens when a small change in a parameter value, not the variables or the equation itself, results in a radical transformation in a systems’ behavior.

Another important concept to chaos theory is periodicity. A mapping has a period 2 cycle if

. The orbit (travel path) is then That is, the point will jump to another value on iteration and then back to the original value on the next iteration; iterations will cause these points to forever jump back and forth between the two values. An -cycle would take iterations to return to the original point; the path or orbit of a point would be defined by the orbit

We can find points in a two-cycle by solving the equation

where the second iteration returns to the original value, after taking an intermediate value .

Then,when

We know that is a solution as it is a fixed point and so is a solution to .

Dividing by yields

. The other two solutions (besides ) to the 4th degree polynomial are found using the quadratic formula and are

is a two-cycle meaning that the orbit of is

When there are no real solutions to the 2-cycle. When the two roots are the same. It is when that the two cycle is born; this is the onset of period doubling cascade which will be discussed later in the paper. This is the bifurcation mentioned above.

To find out where the 2-cycle, is attracting, we use a similar method as we did to find where the fixed points are attracting. We need to determine when

(chain rule)

as is a two cycle)

Find

When

We know that the 2-cycle is born when and that this attracting 2-cycle is the start of the period doubling cascade. The question then is at what value of the parameter will the 4-cycle begin?

To find the beginning of the 4-cycle, we need to locate the intersection of that is, solve the equation, . In order to get rid of the 2-cycle solutions we have found previously (a 2-cycle iterated twice will be a trivial 4-cycle), we divide by the solutions to

Using the Wolfram website, we find that this gives a 12-th degree polynomial. The discriminant of this polynomial is

Whose only real positive roots are and are (Wolfram MathWorld, n.d.)

Therefore, the onset of the 4-cycle, the bifurcation point, is at (

In summary, 0 is an attracting fixed point when and

. We know that is an attracting two-cycle for and that for

Below is a bifurcation diagram, a table and a line graph showing this information each powered by Pynamical software. A bifurcation diagram graphs as the independent variable and as the dependent variable. This type of graph shows the values that the equation is attracted to given a value The bifurcation diagram shows 7 growth values.

Chart

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Pynamical software was used to run the logistic model 20 times for growth rates 0.6, 1.1, 1.6, 2.1, 2.6, 3.1 and 3.5. These growth rates were chosen as they are (nearly) equally spaced and avoid growth rates as the fixed point is indifferent at these values 3.5 is used instead of 3.6 for reasons that will become clear. The model always starts the population level at 0.5.

Table 1. Population values produced by the logistic map with 7 growth rate parameter values over 20 generations.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Generation | r=0.6 | r=1.1 | r=1.6 | r=2.1 | r=2.6 | r=3.1 | r=3.5 |
| 1 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |
| 2 | 0.150 | 0.275 | 0.400 | 0.525 | 0.650 | 0.775 | 0.875 |
| 3 | 0.077 | 0.219 | 0.384 | 0.524 | 0.592 | 0.541 | 0.383 |
| 4 | 0.042 | 0.188 | 0.378 | 0.524 | 0.628 | 0.770 | 0.827 |
| 5 | 0.024 | 0.168 | 0.376 | 0.524 | 0.607 | 0.549 | 0.501 |
| 6 | 0.014 | 0.154 | 0.376 | 0.524 | 0.620 | 0.768 | 0.875 |
| 7 | 0.008 | 0.143 | 0.375 | 0.524 | 0.613 | 0.553 | 0.383 |
| 8 | 0.005 | 0.135 | 0.375 | 0.524 | 0.617 | 0.766 | 0.827 |
| 9 | 0.003 | 0.128 | 0.375 | 0.524 | 0.614 | 0.555 | 0.501 |
| 10 | 0.002 | 0.123 | 0.375 | 0.524 | 0.616 | 0.766 | 0.875 |
| 11 | 0.001 | 0.119 | 0.375 | 0.524 | 0.615 | 0.556 | 0.383 |
| 12 | 0.001 | 0.115 | 0.375 | 0.524 | 0.616 | 0.765 | 0.827 |
| 13 | 0.000 | 0.112 | 0.375 | 0.524 | 0.615 | 0.557 | 0.501 |
| 14 | 0.000 | 0.109 | 0.375 | 0.524 | 0.615 | 0.765 | 0.875 |
| 15 | 0.000 | 0.107 | 0.375 | 0.524 | 0.615 | 0.557 | 0.383 |
| 16 | 0.000 | 0.105 | 0.375 | 0.524 | 0.615 | 0.765 | 0.827 |
| 17 | 0.000 | 0.104 | 0.375 | 0.524 | 0.615 | 0.558 | 0.501 |
| 18 | 0.000 | 0.102 | 0.375 | 0.524 | 0.615 | 0.765 | 0.875 |
| 19 | 0.000 | 0.101 | 0.375 | 0.524 | 0.615 | 0.558 | 0.383 |
| 20 | 0.000 | 0.100 | 0.375 | 0.524 | 0.615 | 0.765 | 0.827 |

Notice that the attracting fixed points and attracting 2 and 4 cycle are the values that the population tends to with iteration.

Recall:

0 is an attracting fixed point when The table shows the population tending to 0 on iteration for

for . The table shows that:

when the population tends toward the attracting fixed point

when the population tends toward the attracting fixed point

when the population tends toward the attracting fixed point

when .6154

We know that is an attracting 2-cycle for

When

which are the values that Table 1 shows the population oscillating between for

When we did not calculate these 4 values but the 4 values that the population tends toward oscillating between are evident using Table 1.

The values the population tends towards is easily seen in the line graph below.

Chart, line chart

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So far, the information that we have found is interesting but shows no chaotic behavior. At this point, the math involved in finding the onset of the next period doubling value is extremely complicated. We will rely again on Wolfram website to find some additional points where bifurcation occurs. For between 3.44949… and 3.54409…, the population will tend toward permanent oscillations among 4 values (a 4-cycle). The as increases past 3.54409 the population will approach oscillations among 8, then 16, then 32 values etc. while the lengths of the intervals get smaller rapidly.

At 3.56995 there is the onset of chaos and the end to the period doubling cascade. From nearly all initial conditions there is no longer oscillations of a finite period.

However, even for 3.56995, there are isolated ranges for that show non-chaotic behavior (Wolfram MathWorld, n.d.).

It is for that the logistic equation exhibits the most popular chaotic behavior, sensitivity to initial conditions known as popularly as the butterfly effect. That is, starting the iteration of the logistic map will result in wildly divergent orbits for .

Scientists have relied on visual and qualitative approaches to investigate the dynamics of the logistic mapping and other examples of nonlinear dynamics; we will too. Below is the bifurcation diagram for 1000 increments of values from [0,4] to show the onset of chaos. The equation has been iterated 200 times but only the last 100 iterations are graphed so as to give the population at the value time to settle to its attractors; this gives a crisper graph so that the system’s attractors as a function of the parameter are easier to see.

1000 vertical slices, one at each value, show the attractor(s) for that value. The value(s) of at each growth value represents 100 generations; for small values or , for instance all 100 iterations are represented by one value, the attractor. For some growth rates ,for instance has 100 different values which never repeat; does not settle to any value or cycle (Boeing, 2016).

Chart

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The bifurcation graph is called self-similar as magnification of r value intervals in the chaotic region, show similar detail to the entire graph (Wolfram MathWorld, n.d.) see below.

The logistic mapping can be used for all types of growth modeling when exponential growth is impeded by death, therapy, saturation or other causes that have a deleterious on otherwise exponential growth. The spread of a virus, rumors, inventions and ideas can be modeled using the equation. For instance, the logistic map is used to model the growth of tumors by modifying the baseline proliferation rate of cancer cells to include the “therapy induced” death rate of cancer cells (Wolfram MathWorld, n.d.).

The chaotic region of the logistic map is used to generate random numbers. Certain applications require higher degrees of randomness and want the random numbers to follow the normal distribution. In these cases, chaotic maps are constructed by modifying known chaotic maps such as the logistic mapping for r> 4. “These modifications usually follow simple techniques, such as introducing

additional nonlinear terms in the system’s differential/difference equations, changing an existing

term to a higher-order term, or even by adding new variables to make the system hyperchaotic” (Moysis et al, 2020). An example of a hyperchaotic equation from Moysis et al is

Where With the modifications shown above, the values of the chaotic map, are multiplied by the value [0,1] computed using the decimal part of [0,1] so that remains in the interval [0,1]

# We have only scratched the surface of the deterministic, yet chaotic logistic mapping. Further reading on the subject should include Robert May’s “Simple Mathematical Models With Very Complicated Dynamics” and Robert Devaney’s “Chaotic Bursts in Nonlinear Dynamical Systems.”

References

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